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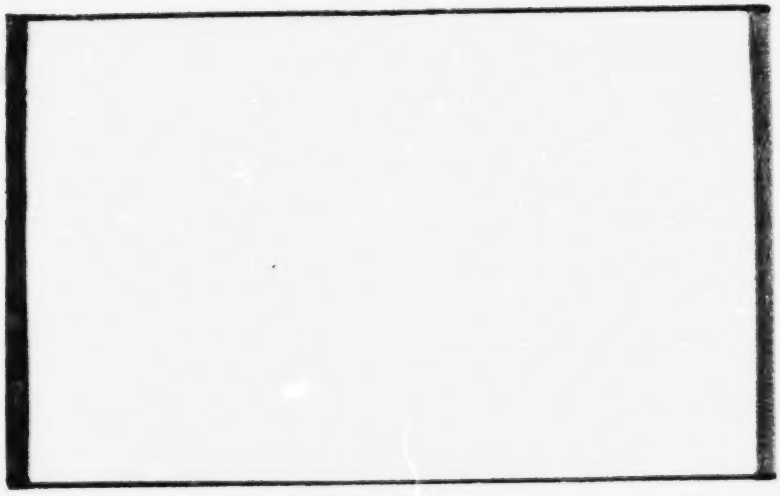

QUEUING MODELS FOR BATCH PROCESSING
Applications to Rail Yard Classification

by

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ABSTRACT

Consider a situation where jobs accumulate at node A to form a batch and then enter a queue for processing at node B. There are two policies for entering the queue at node B - entering at constant intervals or entering in constant batch sizes. It is established that the constant batch size policy performs better than the constant interval policy while minimizing the mean of the accumulation plus processing times, i.e. the mean delay, under the condition that the number of entries be the same for the two policies. If the total cost due to delay and entry is to be minimized, the constant batch policy again performs better. If one would rather minimize the variance of the delay, there is then a trade-off situation between the two policies, with constant interval policy performing better for low arrival rates relative to the service at rate B.

The option of not incurring entry costs under constant interval policy, when the accumulation is nil, is also considered. It is then established that while minimizing total costs (of delay and entry), the constant batch size policy still performs better than the new constant interval policy.

Applications can readily be seen in computer processing, production scheduling, railroad classification, shuttle bus service operation, etc.

Similar problems were considered by Osuna and Newell (1972) for a public transportation system and Petersen (1971a and 1971b) for railroad modelling.

This paper also considers a modified constant interval policy wherein entry costs are not incurred if accumulation is zero in any interval.

Expressions for Expected Delay are developed in Section 2, followed by an evaluation of the policies for minimum expected delay in Section 3. In Section 4, the policies are evaluated under the criteria of minimum expected total cost. As an aside in Section 5, the optimal interval and the optimal batch size that lead to their respective minimum total costs are analyzed as a function of the arrival rate.

Section 6 develops the expressions for variance, followed by the evaluation of the policies for minimum variance in Section 7.

Sections 8 and 9 deal with the modified constant interval policy, developing the optimal interval, the associated minimum cost and comparing it with the other policies.

Section 10 represents the conclusions with some questions for future research.

1. INTRODUCTION

Consider a problem where jobs, arriving in a Poisson process, accumulate at node A to form a batch and then enter a queue for processing at node B, with significant entry costs. There are two policies by which the batches can be entered into the queue - at constant intervals (policy 1) or in constant batch sizes (policy 2). To decide which policy is better, we can evaluate the policies under the criteria of

i) expected delay per job, while maintaining same number of batch entries into B, under either policy;

ii) expected total cost of delay and entry;

iii) variance in delay per job, with same number of entries under either policy.

Delay has two components :

i) delay incurred at A, while forming the batch before entry into the the queue and

ii) delay incurred at B while waiting in the queue for processing.

As an example, in a railroad context, where cars accumulate to form trains, the delay at A would be called the connection delay and the delay at B - the classification delay. The entry cost would be the cost of moving the train from A to B (primarily line-haul costs).

This problem with criteria (i) and (iii) had first been considered by Turnquist and Daskin (1982).

2. EXPRESSIONS FOR EXPECTED DELAY

At node A :

Under policy 1, the expected delay

$$E(T_{1A}) = H/2 \tag{1}$$

where H is the interval headway, since for any arrival, the waiting time is uniformly distributed in the interval [0,H].

Under policy 2, the expected delay

$$E(T_{2A}) = \frac{L-1}{2r} \tag{2}$$

where L is the batch size and r is the rate of arrival of jobs.

To prove (2) : Each arrival faces an Erlang distribution of order k, if k more arrivals are needed to form a batch. So the ith arrival faces an Erlang distribution of order (L-i) since (L-i) more arrivals would complete a batch of size L. Assuming that a job can be any of the arrivals with equal probability, we have :

$$E(T_{2A}) = \sum_{i=1}^{L} \frac{1}{L}(E(T_{2A}) \text{ of ith arrival})$$

$$= \sum_{i=1}^{L} \frac{1}{L} \frac{L-i}{r}$$

$$= \frac{1}{rL}\left[\frac{L(L-1)}{2}\right] = \frac{L-1}{2r} \quad . \qquad \square$$

(2) differs from a similar model proposed by Turnquist and Daskin (1982) in which the authors are in error since they consider an arrival of a car to be an observer to the batch accumulation process rather than a

participant. This then brings out $E(T_{2A})$ as $\frac{L+1}{2r}$ (#67 in their paper), and affects their conclusions.

At node B : We consider a Queue discipline that is FCFS among batches and random within. Then the general expression for the expected waiting time at node B due to Burke (1975) is

$$= \left(\frac{1}{2\mu_B}\right) \left[\left(\frac{1}{1-\rho_B}\right)\left(\frac{L_2}{L_1} + \rho_B \mu_B^2 \sigma_B^2\right) - 1 \right] \qquad (3)$$

where L_1 = first moment of batch size (mean)

L_2 = second moment of batch size (about the origin)

μ_B = mean service rate at B (jobs per unit time)

σ_B^2 = variance of service time distribution

ρ_B = utilization at B = λ_B/μ_B where

λ_B = arrival rate of jobs at B.

We assume a deterministic service time at B, hence $\sigma_B^2 = 0$.

Under policy 1, we have batch sizes having a Poisson distribution with mean rH, and consequently the second moment about the origin is rH(rH+1).

So, $E(T_{1B}) = \frac{1}{2\mu_B} \left[\left(\frac{1}{1-\rho_B}\right)(rH+1) - 1 \right]$

$$= \frac{rH+\rho_B}{2\mu_B(1-\rho_B)} \quad . \qquad (4)$$

Under policy 2, we have constant batch sizes.

So, $L_1 = L$ and $L_2 = L^2$.

So, $E(T_{2B}) = \frac{1}{2\mu_B} \left[\left(\frac{1}{1-\rho_B}\right)(L) - 1 \right] = \frac{L-1+\rho_B}{2\mu_B(1-\rho_B)}$. (5)

3. EVALUATION BY MIN DELAY CRITERIA

Under policy 1, the expected total delay ((1) + (4))

$$= \frac{H}{2} + \frac{rH+\rho_B}{2\mu_B(1-\rho_B)} \tag{6}$$

Under policy 2, the expected total delay ((2) + (5))

$$= \frac{L-1}{2r} + \frac{L-1+\rho_B}{2\mu_B(1-\rho_B)} \tag{7}$$

Since we wish to compare the two expressions (6) and (7) keeping the number of entries the same, we have the condition

$$\frac{1}{H} = \frac{r}{L} = \text{number of entries per unit time.}$$

that is $$L/r = H \tag{8}$$

Substituting this condition in (6) and comparing with (7), we see clearly that policy 2 - the constant batch size policy,does better than policy 1 - the constant interval policy, i.e. (7) < (6).

Substracting (7) from (6) gives us the additional delay incurred by the constant interval policy over the constant batch size policy.

We have $(6) - (7) = \frac{1}{2r} + \frac{1}{2\mu_B(1-\rho_B)}$

We know $\rho_B = \lambda_B/\mu_B$. Let $\lambda_B = r + \lambda_0$ where λ_0 are the other jobs (apart from arrivals from A) arriving into B. The additional delay can then be expressed as

$$= \frac{1}{2r} + \frac{1}{2(\mu_B - \lambda_0 - r)}$$

If we call $(\mu_B - \lambda_0)$ as μ_0 which is equal to the net service rate available at B for jobs from A, then the additional delay is expressible as

$$= \frac{1}{2r} + \frac{1}{2(\mu_0 - r)} \tag{9}$$

The first part of (9) is due to node A and the second due to node B. Adding them out yields

$$\frac{\mu_0}{2r(\mu_0 - r)} \tag{10}$$

This expression reaches a minimum of $2/\mu_0$ when $r = \mu_0/2$. As $r \to 0$ and as $r \to \mu_0$, the additional delay becomes large, i.e. the constant batch size policy becomes relatively more favorable (Fig. 1).

If $\mu_0 \to \infty$, i.e. we have node B with infinite service rate, an example being the pedestrian crossing problem as in Larson and Odoni (1981), then (10) reduces to $\frac{1}{2r}$ and consequently the additional delay reduces as r becomes large (Fig. 2).

4. EVALUATION BY MIN COST CRITERIA

To apply the second criteria, for either policy, we first determine the optimal strategy which minimizes total costs per unit time, given as the sum or delay cost and entry cost per unit time and then compare the two policies. The cost of delay is C_j \$ per unit time per job and cost of entry is C_b \$ per entry per batch.

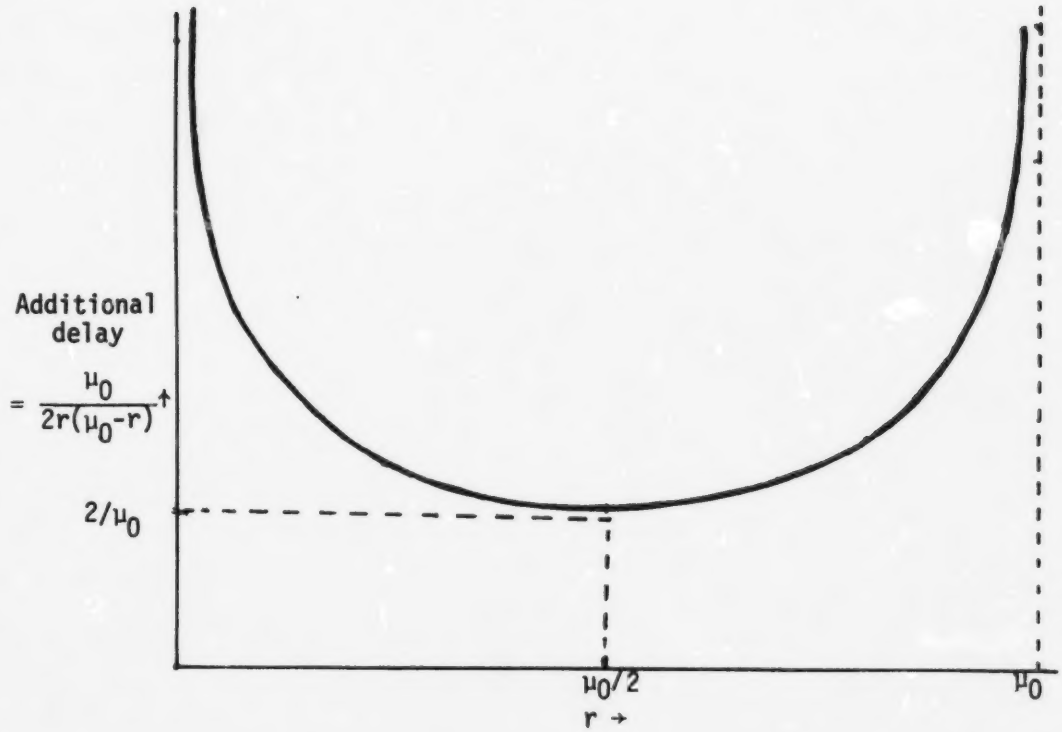

Figure 1 : Additional delay due to Policy 1 over Policy 2 for finite μ_0.

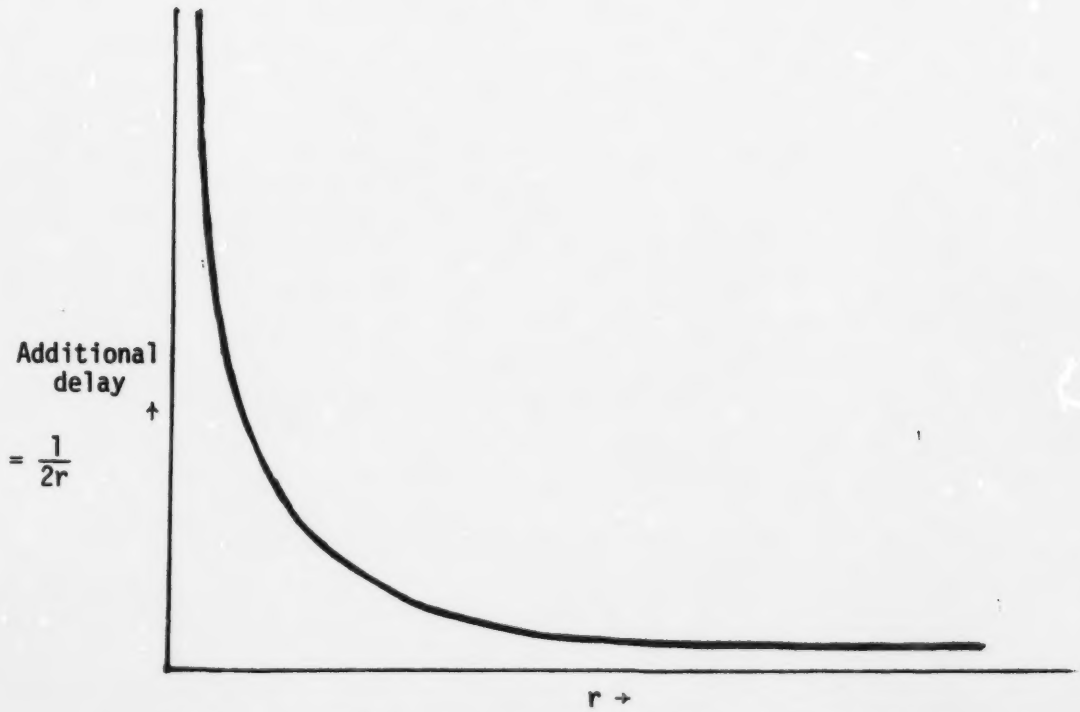

Figure 2 : Additional delay due to Policy 1 over Policy 2 for infinite μ_0.

Under policy 1, the total costs per unit time for an expected arrival rate r

$$= C_j(E(T_{1A}) + E(T_{1B}))r + C_b/H$$

$$= C_j \frac{H}{2} r + C_j \frac{(rH+\rho_B)}{2\mu_B(1-\rho_B)} r + \frac{C_b}{H} \qquad (11)$$

Differentiating with respect to H and setting the derivative = 0,

$$\frac{C_j r}{2} + \frac{C_j r^2}{2\mu_B(1-\rho_B)} - \frac{C_b}{H^{*2}} = 0$$

$$\text{Optimal } H^* = \sqrt{\frac{2C_b\mu_B(1-\rho_B)}{C_j r\,[\mu_B(1-\rho_B)+r]}} \qquad (12)$$

Under policy 2, the total costs per unit time

$$= C_j(E(T_{2A}) + E(T_{2B}))r + \frac{r}{L}$$

$$= C_j\frac{L-1}{2r} \cdot r + C_j\left(\frac{L-1+\rho_B}{2\mu_B(1-\rho_B)}\right)r + C_b\frac{r}{L} \qquad (13)$$

Differentiating with respect to L and setting the derivative = 0,

$$\frac{C_j}{2} + \frac{C_j r}{2\mu_B(1-\rho_B)} - \frac{C_b r}{L^{*2}} = 0$$

$$\text{Optimal } L^* = \sqrt{\frac{2C_b r\mu_B(1-\rho_B)}{C_j\,[\mu_B(1-\rho_B)+r]}} \qquad (14)$$

From equations (12) and (14), we see that the optimal strategies are related as

$$L^* = rH^* \qquad (15)$$

But this is the same as condition (8). So, even while minimizing total costs, the solution requires us to maintain the same number of entries into B in either policy. It follows then from the argument for the previous criteria that the constant batch size policy does better than the constant interval policy since entry costs are the same and delay costs are less for the constant batch size policy. The result could also have been obtained directly as a corollary to the result that (6) > (7) when L = rH (section 3).

Lemma 1. Let $(6) = DF_H(H)$ $(8) = DF_L(L)$ where DF is delay function
Let $(11) = TC_H(H)$ $(13) = TC_L(L)$ where TC is total cost.

If $DF_H(H) > DF_L(L)$ when $L = rH$, then $TC_H(H^*) > TC_L(L^*)$ where H^* and L^* are optimal arguements for TC_H and TC_L respectively.

Proof. We have $DF_H(H) > DF_L(L) \geq 0$ when $L = rH$

Hence for every $C_j > 0$

$$C_j DF_H(H) > C_j DF_L(L)$$

$$\Leftrightarrow C_j DF_H(H) + \frac{C_b}{H} > C_j DF_L(L) + C_b \frac{r}{L}$$

$$\Leftrightarrow TC_H(H) > TC_L(L)$$

$$\Rightarrow TC_H(H^*) > TC_L(rH^*)$$

But, by definition

$$TC_L(rH^*) \geq TC_L(L^*)$$

Hence $TC_H(H^*) > TC_L(L^*)$ □

The total additional cost per unit time due to the constant interval policy is given by (11) - (13) as

$$= \frac{C_j}{2} + \frac{C_j r}{2\mu_B(1-\rho_B)}$$

simplified as

$$= \frac{C_j \mu_0}{2(\mu_0 - r)} \qquad (16)$$

The entry costs thus do not affect the difference in costs. We can observe, higher the value of r with respect to μ_0, higher the additional cost due to the constant interval policy.

The additional cost per job due to the constant interval policy is

$$= \frac{C_j \mu_0}{2r(\mu_0 - r)} \qquad (17)$$

This is analogous to (1) and can be analyzed as in section 3.

5. AN ASIDE

An interesting question on the aside can be : how do the optimal interval and batch size vary with the arrival rate?

Equations (12) and (14) can be rewritten as

$$H^* = \sqrt{\frac{2C_b}{C_j \mu_0}} \quad \sqrt{\frac{\mu_0}{r} - 1} \qquad (18)$$

and

$$L^* = \sqrt{\frac{2C_b}{C_j \mu_0}} \quad \sqrt{r\mu_0 - r^2} \qquad (19)$$

As $r \to 0$, $H^* \to \infty$, $L^* \to 0$

$$\frac{dH^*}{dr} \to -\infty \ , \quad \frac{dL^*}{dr} \to \infty \quad .$$

As $r \to \mu_0$ $H^* \to 0$, $L^* \to 0$

$$\frac{dH^*}{dr} \to -\infty, \quad \frac{dL^*}{dr} \to -\infty \quad .$$

There is a point of inflexion at $r = \frac{3\mu_0}{4}$ for H^* (Fig. 3)

and a point of maximum at $r = \frac{\mu_0}{2}$ for L^* . (Fig. 4)

The optimal interval is continuously decreasing, since there are enough savings in the delay costs at A and B to compensate for the increase in entry costs. The optimal batch size is increasing at first since savings in entry costs compensate for delay costs. When $r > \mu_0/2$. the congestion effects at B become large enough that it is better to send in smaller batches, though more frequently.

6. EXPRESSIONS FOR VARIANCE

At node A :

Under policy 1, the variance

$$V(T_{1A}) = H^2/12 \qquad (20)$$

because for any arrival, the waiting time is uniformly distributed in the interval [0,H].

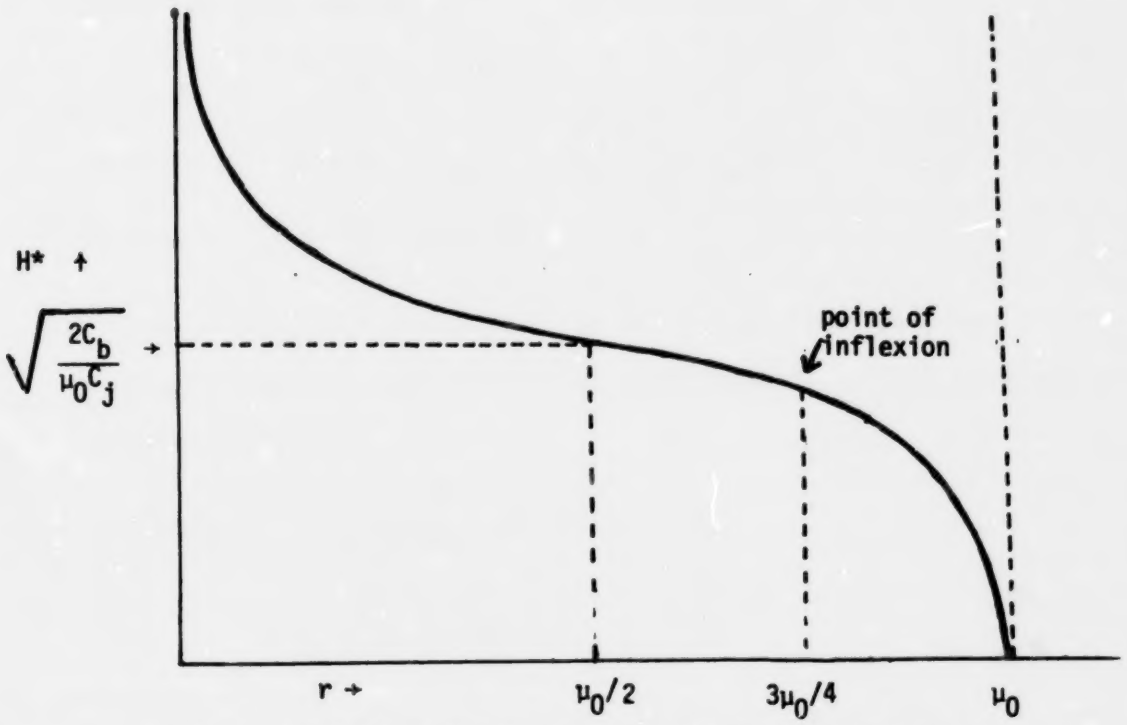

Figure 3 : Optimal headway as a function of arrival rate.

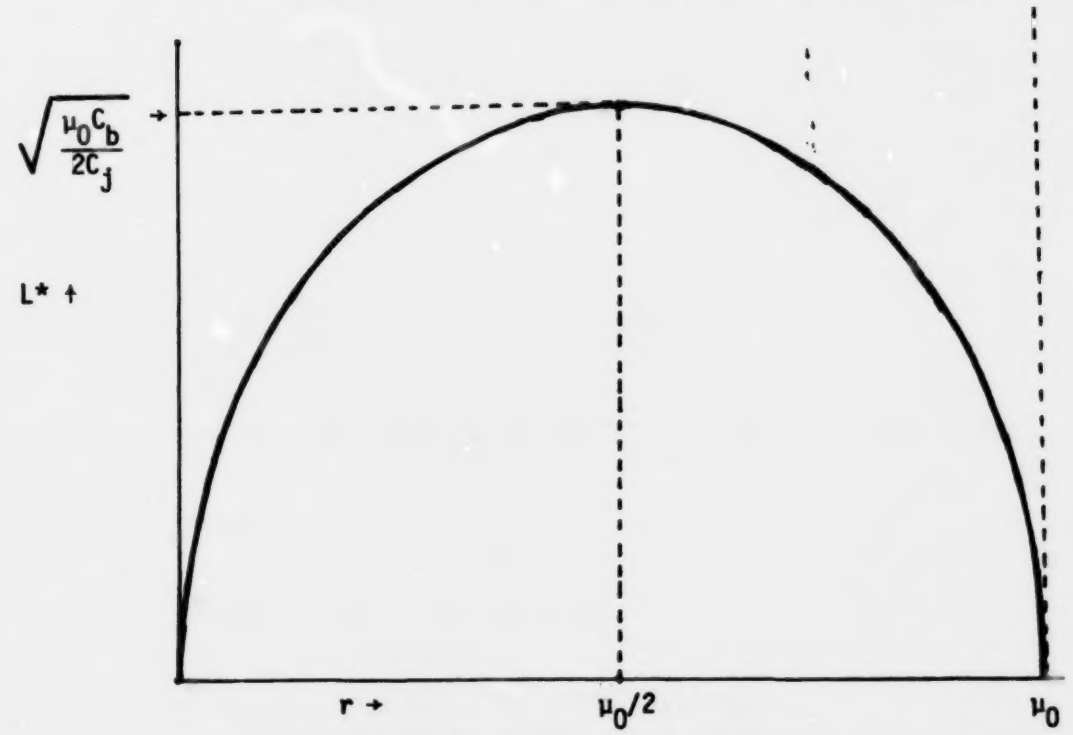

Figure 4 : Optimal batchsize as a function of arrival rate.

Under policy 2, the variance

$$V(T_{2A}) = \frac{(L+7)(L-1)}{12r^2} \tag{21}$$

To prove (21) we use $V(T_{2A}) = E(T_{2A}^2) - [E(T_{2A})]^2$.

We know $E(T_{2A}) = \frac{L-1}{2r}$ as in (2)

To obtain $E(T_{2A}^2)$, we notice that the ith arrival faces an Erlang distribution of order L-i, and there is an equal probability $\frac{1}{L}$ of being any of the arrivals.

So $E(T_{2A}^2) = \sum_{i=1}^{L} \frac{1}{L} (E(T_{2A}^2)$ of ith arrival)

$$= \sum_{i=1}^{L} \frac{1}{L} \cdot \frac{(L-i)(L-i+1)}{r^2}$$

$$= \frac{1}{Lr^2} \sum_{j=1}^{L-1} j(j+1)$$

$$= \frac{1}{Lr^2} \left[\frac{(L-1)(L)(2L-1)}{6} + \frac{L(L-1)}{2} \right]$$

$$= \frac{L^2-1}{3r^2}$$

$$V(T_{2A}) = \frac{L^2-1}{3r^2} - \frac{(L-1)^2}{4r^2} = \frac{(L+7)(L-1)}{12r^2} \qquad \square$$

(21) is a corrected version of #66 in Turnquist and Daskin (1982).

At node B :

Under policy 1, the variance

$$V(T_{1B}) = \frac{(2\rho_B+1)(rH)^2 + 6rH + \rho_B(4-\rho_B)}{12\mu_B^2(1-\rho_B)^2} \tag{22}$$

Under policy 2, the variance

$$V(T_{2B}) = \frac{(2\rho_B+1)(L)^2 - (1-\rho_B)^2}{12\mu_B^2(1-\rho_B)^2} \qquad (23)$$

The derivation of (22) and (23) can be seen in Turnquist and Daskin (1982).

7. EVALUATION BY MIN VARIANCE CRITERIA

Assuming independence at A and B, we can add the variances and find the difference of total variance between policies 1 and 2.

So, $[V(T_{1A}) + V(T_{1B})] - [V(T_{2A}) + V(T_{2B})]$

$$= \frac{H^2}{12} + \frac{(2\rho_B+1)(rH)^2 + 6rH + \rho_B(4-\rho_B)}{12\mu_B^2(1-\rho_B)^2}$$

$$- \frac{(L+7)(L-1)}{12r^2} - \frac{(2\rho_B+1)L^2-(1-\rho_B)^2}{12\mu_B^2(1-\rho_B)^2}$$

Substituting $H = L/r$

$$= \frac{7-6L}{12r^2} + \frac{6L+4\rho_B-\rho_B^2+1+\rho_B^2-2\rho_B}{12\mu_B^2(1-\rho_B)^2}$$

$$= \frac{7-6L}{12r^2} + \frac{6L+2\rho_B+1}{12\mu_B^2(1-\rho_B)^2} \qquad (24)$$

The first term which is negative as long as $L > 1$, implies that the variance at A is better (lesser) for the constant interval policy. The second term which is positive implies that the variance at B is always lesser for constant batch size policy.

We wish to determine if the whole expression (24) is always greater than or less than zero, or if there is a trade-off situation between the two policies.

Solving (24) further, we have :

Variance due to constant interval policy is greater than variance due to constant batch size policy if and only if

$$(7-6L)\mu_B^2(1-\rho_B)^2 + (6L+2\rho_B+1)r^2 \geq 0$$

$$\Leftrightarrow \quad r^2 \geq \frac{(6L-7)\mu_B^2(1-\rho_B)^2}{6L+2\rho_B+1}$$

If L=1, the right hand side is negative and we have the result that constant batch size policy has a lower variance always (for any level of arrival rate r).

If $L > 1$,

$$\Leftrightarrow r \geq \sqrt{\frac{6L-7}{6L+2\rho_B+1}} \quad \mu_B(1-\rho_B) \ .$$

Since $\rho_B = \frac{\lambda_B}{\mu_B}$, $\lambda_B = \lambda_0 + r$ and $\mu_0 = \mu_B - \lambda_0$

$$\Leftrightarrow r \geq \sqrt{\frac{6L-7}{6L+2\rho_B+1}} \quad (\mu_0 - r) \tag{25}$$

Since $\rho_B = \frac{\lambda_0+r}{\mu_B}$ and complicates the evaluation of r, we write $6L+2\rho_B+1$ as $6L+2+2(\rho_B - \frac{1}{2})$ and neglect $2(\rho_B - \frac{1}{2})$. The maximum absolute error is 1 (at ρ_B=0 or 1). The maximum relative error is 7.2% with respect to a minimum value of 14 when L = 2. For any other value of ρ_B or L, the relative error is less.

So we write (25) as :

$$\Leftrightarrow r \geq K(\mu_0 - r)$$

where

$$K = \sqrt{\frac{6L-7}{6L+2}}$$

$$\Leftrightarrow r \geq \frac{K\mu_0}{1+K} \qquad (26)$$

As L becomes large, K/(1+K) tends to 1/2 (Table 1).

TABLE 1

L=rH	K	K/(1+K)
2	0.60	0.37
3	0.74	0.42
4	0.81	0.45
5	0.85	0.46
⋮		
10	0.92	0.48
20	0.96	0.49

For large L, we approximate K/(1+K) to 1/2 and derive the condition that

$$\Leftrightarrow r \geq \frac{\mu_0}{2} \quad . \qquad (26)$$

If r is greater than half the available service rate at B, we can say definitely that the constant batch size policy has lesser variance (Fig. 5).

If $\mu_0 \to \infty$, i.e. we have processing at B with infinite service rate as in the pedestrian crossing problem mentioned in Larson and Odoni (1981), then the constant interval policy always has lesser variance, except when L = 1.

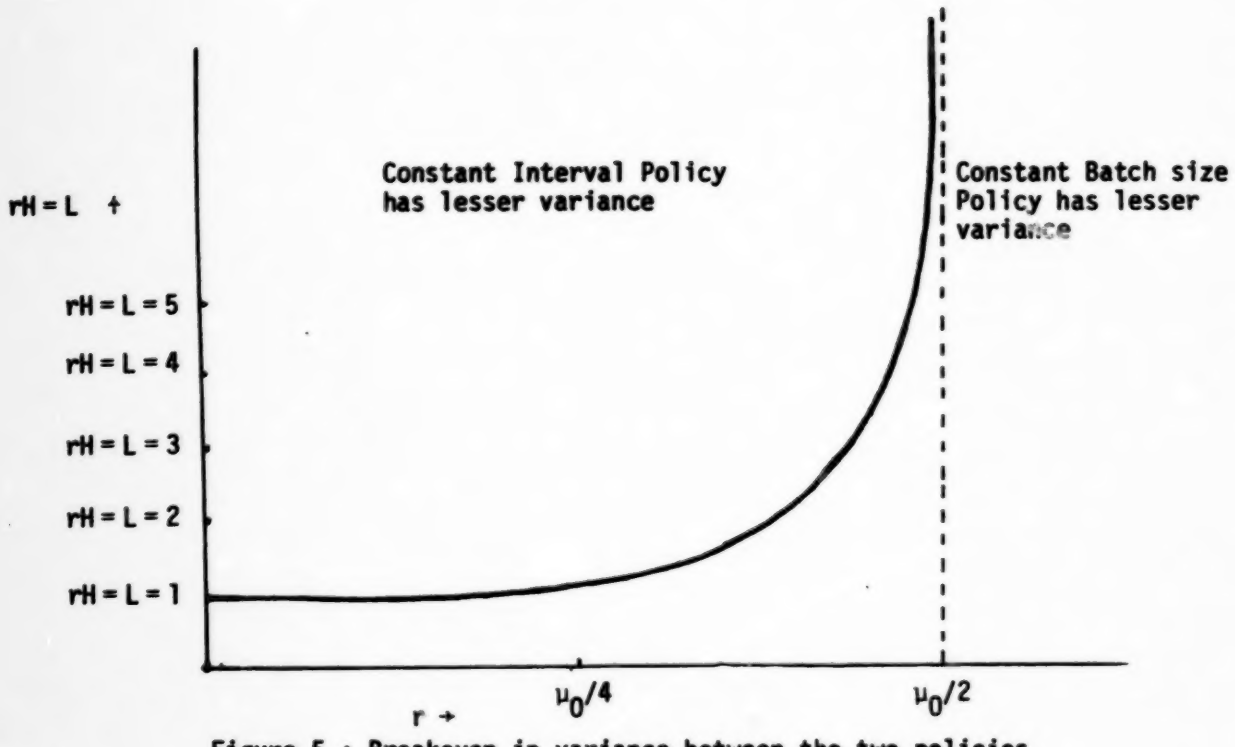

Figure 5 : Breakeven in variance between the two policies

In this analysis, we assume that L or H is set externally. If it is set by a known relation as a function of r, like (18) or (19), then by substituting the same into (25), we can obtain one (complicated!) expression relating r to C_b, C_j, μ_0 and λ_0, that expresses the breakeven variance condition for the two policies.

8. A MODIFIED CONSTANT INTERVAL POLICY - CALCULATION OF OPTIMAL INTERVAL

We modify the Constant Interval Policy, considering that entry costs are not incurred when there are no arrivals in an interval.

The probability of no arrivals in an interval $H = e^{-rH}$.

So the expected number of entries made in unit time

$$= (1-e^{-rH}) \cdot \frac{1}{H} \quad . \qquad (27)$$

The optimal interval is now to be determined from the total cost which is similar to (11) :

$$= C_j \frac{Hr}{2} + C_j \frac{(rH+\rho_B)}{2\mu_B(1-\rho_B)} r + \frac{C_b}{H}(1-e^{-rH}) \qquad (28)$$

We cannot obtain a closed form expression for the optimal $\bar{H}$ from (28). Nevertheless, we can analyze the situation by considering the point where the slope of the increasing delay costs are equal to the slope of the decreasing entry costs (Fig. 6).

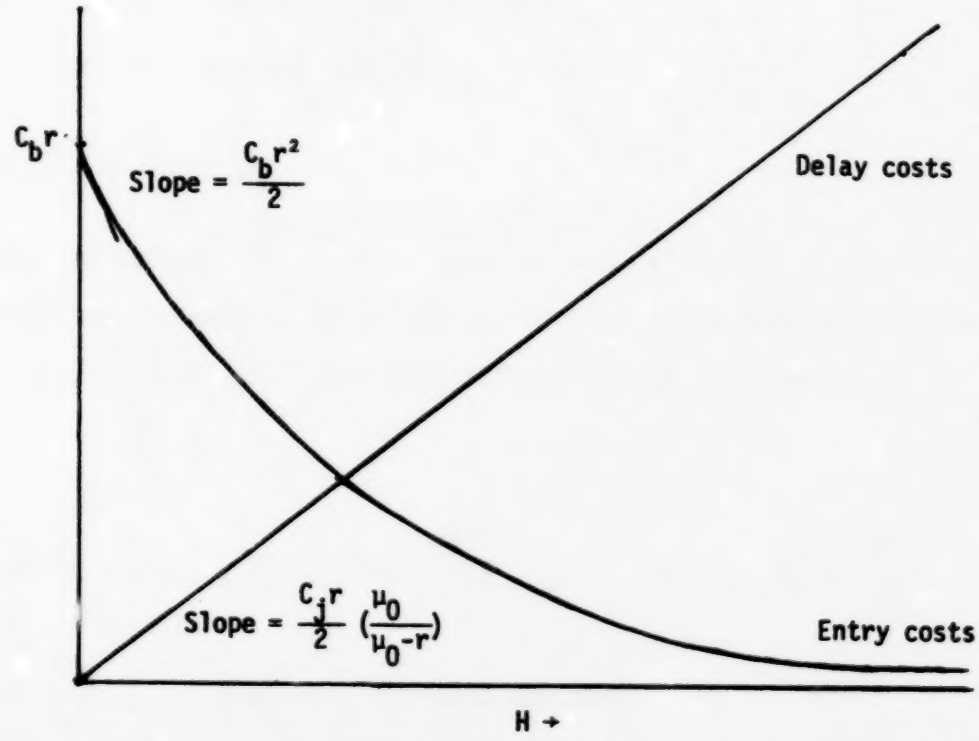

Figure 6 : Cost Curves for the Modified Constant Interval Policy.

The slope of the delay costs

$$= \frac{C_j r}{2} + \frac{C_j r^2}{2\mu_B(1-\rho_B)}$$

$$= \frac{C_j r}{2} \left(1 + \frac{r}{\mu_0 - r}\right)$$

$$= \frac{C_j r}{2} \left(\frac{\mu_0}{\mu_0 - r}\right) \tag{29}$$

The slope of the entry costs

$$= C_b \frac{e^{-rH}(1+rH)-1}{H^2} \tag{30}$$

As $H \to 0$, the above slope $\to -\frac{C_b r^2}{2}$

$H \to \infty$, the above slope $\to 0$

If $\frac{C_j r}{2} \left(\frac{\mu_0}{\mu_0 - r}\right) \geq \frac{C_b r^2}{2}$, (31)

then the optimal $\overline{H} = 0$, (i.e. enter as soon as we have an arrival) since the increase in delay cost can never be compensated by a decrease in entry costs. The above condition can be stated further as :

$$r^2 - \mu_0 r + \frac{C_j \mu_0}{C_b} \geq 0 \tag{32}$$

$$\Leftrightarrow r \leq \frac{\mu_0}{2}\left(1 - \sqrt{1 - \frac{4C_j}{\mu_0 C_b}}\right) \text{ or } r \geq \frac{\mu_0}{2}\left(1 + \sqrt{1 - \frac{4C_j}{\mu_0 C_b}}\right)$$

If $\frac{4C_j}{\mu_0 C_b} > 1$, then (32) is always satisfied.

Assuming (32) is not satisfied, i.e. $\frac{4C_j}{\mu_0 C_b} < 1$ and

$$\frac{\mu_0}{2}\left(1-\sqrt{1-\frac{4C_j}{\mu_0 C_b}}\right) < r < \frac{\mu_0}{2}\left(1+\sqrt{1-\frac{4C_j}{\mu_0 C_b}}\right)$$

then optimal $\bar{H} > 0$.

To obtain the value, we need to set (29) + (30) = 0 and solve for $\bar{H}$ numerically.

Then $$\frac{C_j r}{2}\left(\frac{\mu_0}{\mu_0 - r}\right) + C_b \frac{e^{-r\bar{H}}(1+r\bar{H})-1}{\bar{H}^2} = 0$$

$$\Leftrightarrow \quad \frac{C_j}{2C_b r}\left(\frac{\mu_0}{\mu_0 - r}\right) = \frac{1-e^{-r\bar{H}}(1+r\bar{H})}{\bar{H}^2}$$

Let $$C = \frac{C_j \mu_0}{2C_b r(\mu_0 - r)} \tag{33}$$

and $\bar{X} = r\bar{H}$

Then $$C = \frac{1-e^{-\bar{X}}(1+\bar{X})}{\bar{X}^2} \tag{34}$$

We solve for $\bar{X}$ numerically and obtain $\bar{H} = \frac{\bar{X}}{r}$

Either $\bar{X} = e^{\bar{X}}(1-C\bar{X}^2)-1$ or $\bar{X} = \log\left[\frac{1+\bar{X}}{1-C\bar{X}^2}\right]$ can be tried recursively. Convergence is not guaranteed, but a trial and error approach within $0 < \bar{X} < \sqrt{\frac{1}{C}}$ should work.

The critical value of C, above which $\bar{X}$ takes a value zero, is $\frac{1}{2}$ i.e. if $C = \frac{C_j}{2C_b r}\left(\frac{\mu_0}{\mu_0 - r}\right) \geq \frac{1}{2}$, then (31) is satisfied and $\bar{H} = 0$.

Solving (34), we have values for $\overline{X} = r\overline{H}$ as in Table 2.

It can be established that $\frac{dc}{d\overline{X}} < 0$ and $0 \le \overline{X} \le \sqrt{\frac{1}{C}}$

TABLE 2

C	$\overline{X} = r\overline{H}$
1/2	0
1/4	1.09
1/16	3.78

The optimal total cost now is expressed as

$$TC_X(\overline{X}) = C_j \frac{\overline{X}}{2} + C_j \left(\frac{\overline{X}+\rho_B}{2\mu_B(1-\rho_B)}\right) r + \frac{C_b r}{\overline{X}} (1-e^{-\overline{X}})$$

Substituting (33) to eliminate C_b,

$$TC_X(\overline{X}) = C_j \frac{\overline{X}}{2} + \frac{C_j \overline{X} r}{2(\mu_0 - r)} \frac{C_j}{2C} \frac{\mu_0}{\mu_0 - r} \frac{(1-e^{-\overline{X}})}{\overline{X}} + \frac{C_j \rho_B r}{2(\mu_0 - r)}$$

$$= C_j \frac{\mu_0}{\mu_0 - r} \left(\frac{\overline{X}}{2} + \frac{(1-e^{-\overline{X}})}{2C\overline{X}}\right) + \frac{C_j}{2} \frac{\rho_B r}{(\mu_0 - r)} \qquad (35)$$

9. EVALUATION OF MODIFIED CONSTANT INTERVAL POLICY - MIN COST CRITERIA

For the sake of comparison, we develop the optimal total cost expressions for the constant interval policy (with entry costs incurred everytime) and the constant batch size policy.

From (11)

$$TC_H(H^*) = C_j \frac{H^* r}{2} + C_j \frac{(rH^* + \rho_B) r}{2\mu_B(1-\rho_B)} + \frac{C_b}{H^*}$$

Substituting (18) for H* and simplifying,

$$TC_H(H^*) = \sqrt{\frac{2C_jC_br\mu_0}{\mu_0-r}} + \frac{C_j\rho_Br}{2(\mu_0-r)}$$

Substituting (33) to eliminate C_b,

$$TC_H(H^*) = \frac{C_j\mu_0}{\mu_0-r}\sqrt{\frac{1}{C}} + \frac{C_j\rho_Br}{2(\mu_0-r)} \tag{36}$$

From (13)

$$TC_L(L^*) = \frac{C_j(L^*-1)}{2} + \frac{C_j(L^*-1+\rho_B)r}{2\mu_B(1-\rho_B)} + \frac{C_br}{L^*}$$

Substituting (19) for L* and simplifying,

$$TC_L(L^*) = \sqrt{\frac{2C_jC_br\mu_0}{\mu_0-r}} - \frac{C_j\mu_0}{2(\mu_0-r)} + \frac{C_j\rho_Br}{2(\mu_0-r)}$$

Substituting (33) to eliminate C_b,

$$TC_L(L^*) = \frac{C_j\mu_0}{\mu_0-r}\left(\sqrt{\frac{1}{C}} - \frac{1}{2}\right) + \frac{C_j\rho_Br}{2(\mu_0-r)} \tag{37}$$

The performance of the modified constant interval policy can be determined by comparing $TC_X(\bar{X})$ with respect to $TC_H(H^*)$ and $TC_L(L^*)$.

Lemma 2. $TC_H(H^*) \geq TC_X(\bar{X})$. (38)

Proof. The logic is similar to Lemma 1.

By comparing (11) and (28), we have :

$$TC_H(H^*) \geq TC_X(X=rH^*) .$$

But by definition

$$TC_X(X=rH^*) \geq TC_X(\overline{X})$$

Hence $$TC_H(H^*) \geq TC_X(\overline{X})$$ □

Next, it can be established that $TC_X(\overline{X}) \geq TC_L(L^*)$ (39)

by proving that (35) ≥ (37), i.e. we need to prove that

$$\frac{\overline{X}}{2} + \frac{(1-e^{-\overline{X}})}{2C\overline{X}} \geq \sqrt{\frac{1}{C}} - \frac{1}{2} \qquad \text{for } \overline{X} \geq 0 \ . \tag{40}$$

Substituting (34) for C and using the series expansion of $e^{-\overline{X}}$, we can establish (40) by some messy algebra.

Combining (38) and (39), we have :

$$TC_H(H^*) \geq TC_X(\overline{X}) \geq TC_L(L^*) \ .$$

It thus established that the optimal cost for the constant batch size policy is still better than the constant interval policy, even if entry costs are not incurred when accumulation is nil.

We note that the analysis of variance is not affected by the modified constant interval policy, since variance is purely a function of the arrival process and our "willingness" to make entries at a constant interval. So the analysis of variance as in sections 6 and 7 are valid here too.

As $C \to 1$, the modified constant interval policy performs like the constant batch size policy and as $C \to 0$, it performs like a regular constant interval policy.

10. CONCLUSIONS AND QUESTIONS FOR FUTURE STUDY

Conclusions :

While comparing the Constant Interval Policy and the Constant Batch Size Policy, we have the following results from this paper :

i) Under the criteria of maximum expected delay per job, while maintaining the same number of batch entries from A to B for either policy, the constant batch size policy does better than the constant interval policy.

ii) Under the criteria of minimum expected total cost of delay and entry, constant batch size policy does better than constant interval policy.

iii) Under the criteria of minimum variance in delay per job,

- If batch size = 1, constant batch size policy does better.
- If batch size is larger, constant batch size policy does better if the arrival rate is greater than half the net available service rate at B for arrivals from A. If the arrival rate is not as much, then we need to verify using Table I as to which policy does better. For low arrival rates with large batch sizes, constant interval policy would perform better (as would be justified by intuition).

While comparing the modified constant interval policy (i.e., if accumulation is nil in an interval, we do not incur entry costs) with the other two policies, we have

i) Under the criteria of minimum expected total cost of delay and entry, the constant batch size policy does better than the modified constant interval policy which obviously does better than the constant interval policy.

ii) Under the criteria of minimum expected delay per job or minimum variance in delay per job, the modified constant interval policy performs exactly as the constant interval policy does.

Questions for the future :

- Can we determine an optimal number of arrivals below which a "no entry option" should be examined for the constant interval policy (rather than 1 as in the modified constant interval policy)?

- Can we consider mixed strategies for service like
  . If number of arrivals reaches L_{max} before a time interval H, enter L_{max}, set clock to zero.
  . If not, at H, enter all arrivals if number of arrivals is more than L_{min} and set clock to zero.
  . If at H we do not have L_{min}, just set clock to zero, with no entries.
- A variation of the above can be where we do not set clock to zero but to continue working towards 2H, 3H, etc. (ideal when we have announced schedules).

REFERENCES

BURKE, P.J., Delays in single-server queues with batch input. Operations Research 23, 830-833 (1975).

LARSON, R.C. and ODONI, A.R., Urban Operations Research, Prentice-Hall, New Jersey (1981).

OSUNA, E.E. and NEWELL, G.F., Control strategies for an idealized public transportation system. Transportation Science 6, 52-72 (1972).

PETERSEN, E.R., Bulk Service Queues : with Applications to Train Assembly Times". Working Paper 71-2, Queen's University, School of Business, Kingston, Ontario, Canada (1971a).

PETERSEN, E.R., Bulk Queues with Random Batch Size : with Application to Railroad Modelling. Working Paper 71-3, Queen's University, School of Business, Kingston, Ontario, Canada (1971b).

TURNQUIST, M.A. and DASKIN, M.S., Queuing models of classification and connection delay in railyards. Transportation Science 16, 207-230 (1982).